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The ideal conductor limit

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Abstract. This paper compares two methods of statistical mechanics used to study a classical Coulomb system S near an ideal conductor C. The first method consists in neglecting the thermal fluctuations in the conductor C and constrains the electric potential to be constant on it. In the second method the conductor C is considered as a conducting Coulomb system the charge correlation length of which goes to zero. It has been noticed in the past, in particular cases, that the two methods yield the same results for the particle densities and correlations in S. It is shown that this is true in general for the quantities which depend only on the degrees of freedom of S, but that some other quantities, especially the electric potential correlations and the stress tensor, are different in the two approaches. In spite of this the two methods give the same electric forces exerted on S.

1. Introduction

In the equilibrium statistical mechanics of classical Coulomb systems (for instance electrolytes), sometimes one is led to assume that some wall is an ideal conductor (for instance for mimicking an electrode). Two methods have been used to deal with a classical (i.e. non-quantum) Coulomb system S near an ideal conductor C. The first one is to consider from the beginning that the conductor C is ideal and take this into account by constraining the electric potential to be constant on C [1–3]. The second method is to treat the conductor C as a genuine Coulomb system with a microscopic structure and take the limit of zero correlation length [4, 5]; indeed, in that limit, the statistical average of the charge density on the conductor C becomes a surface charge density of zero thickness, a characteristic feature of ideal conductors.

Both methods gave the same results for some quantities in the Coulomb system S, such as the particle densities, and even the fluctuations of these densities as described by the particle correlation functions. The reason for this agreement about some fluctuations is not obvious. In the first approach, there are no fluctuations inside the conductor C. In the second approach, there are thermal fluctuations inside the conductor C; for instance, the potential/potential correlation function has a universal simple form [6] (in three dimensions, $k_{\mathrm{B}}T$ divided by the distance) for distances large compared to the microscopic scale and these

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correlations do not disappear as the zero correlation length limit is taken. However, this seems to have no influence on some quantities in the Coulomb system S. Nevertheless we shall show that some other quantities in S—for instance the electric potential correlations—are different depending on the method used to compute them.

The aim of the present paper is to discuss the relationship between the two approaches. We first treat a simple two-dimensional example in section 2, where we consider two parallel lines; one is the ideal conductor C and the other is the Coulomb system S. The two methods are worked out to find the electric potential and charge correlations and the stress tensor, and we compare the results. In section 3, we treat the general case, look at quantities such as the partition function, the correlations and the stress tensor, and again we compare the results obtained through the two methods.

The reader may either look first at section 2, or go directly to section 3.

2. A simple two-dimensional example

2.1. The model

For simplicity, we consider a system of 'restricted dimension' [7]. The Coulomb interaction between two point-charges q and q' at a distance r from each other has the two-dimensional form $-qq' \ln r$, but the particles are constrained to live on one-dimensional lines. In the plane xOy, the line $y = 0$ is the conductor C at zero potential, while the Coulomb system S lives on the line $y = W$ ($W > 0$).

We consider two cases: either the conductor C is an ideal one from the beginning, or it is the high-density limit of a Coulomb system (in that limit, the microscopic scale goes to zero). We compute the potential and charge correlations and the average stress tensor in each case. These quantities can be obtained by a macroscopic approach [6], using linear response theory and a conducting behaviour assumption. Alternatively, exact microscopic results can be derived in a special model.

A position $\boldsymbol{r} = (x, y)$ is conveniently represented by the complex number $z = x + \mathrm{i}y$.

2.2. The ideal conductor approach

The line $y = 0$ is assumed to be a grounded ideal conductor C, in that sense that the electric potential is constrained to vanish on that line. Thus, in the region $y, y' > 0$, the potential at z' due to a unit point-charge at z is

$$G(\boldsymbol{r}, \boldsymbol{r}') = -\ln \left| \frac{z - z'}{z - \bar{z}'} \right| \tag{2.1}$$

which does vanish on the line $y = 0$. A Coulomb system S lies on the line $y = W$. This model has already been studied [7]. Under the assumption that S is a conductor, a macroscopic approach [6] allows computation of the correlation function for the electric potential Φ at two points $\boldsymbol{r}$ and $\boldsymbol{r}'$ (provided W is macroscopic and $\boldsymbol{r}$ and $\boldsymbol{r}'$ are at macroscopic distances from the line $y = W$). These correlations are

$$\beta \langle \Phi(\boldsymbol{r}) \Phi(\boldsymbol{r}') \rangle^{\mathrm{T}} = -\ln \left| \frac{z - z'}{z - \bar{z}'} \, \frac{\sinh \frac{\pi}{2W}(z - \bar{z}')}{\sinh \frac{\pi}{2W}(z - z')} \right| \qquad \text{if} \quad (y, y') \in]0, W[^2 \tag{2.2a}$$

$$\beta \langle \Phi(\boldsymbol{r}) \Phi(\boldsymbol{r}') \rangle^{\mathrm{T}} = -\ln \left| \frac{z - z'}{z - \bar{z}'} \right| \qquad \text{if} \quad (y, y') \in]0, W[\times]W, +\infty[\tag{2.2b}$$

$$\beta \langle \Phi(\boldsymbol{r}) \Phi(\boldsymbol{r}') \rangle^{\mathrm{T}} = -\ln \left| \frac{z - \bar{z}' - \mathrm{i}W}{z - \bar{z}'} \right| \qquad \text{if} \quad (y, y') \in]W, +\infty[^2. \tag{2.2c}$$

($\langle \ldots \rangle^{\mathrm{T}}$ means a truncated statistical average and β is the inverse temperature.) In the region $y \leqslant 0$, $\Phi(\boldsymbol{r}) = 0$ without fluctuations.

From (2.2), one can compute the correlations of the electric field $\boldsymbol{E}(\boldsymbol{r})$, and from the discontinuities of $E_y(\boldsymbol{r})$ on the lines $y = 0$ and $y = W$ one obtains the correlations for the charge densities $\sigma(\boldsymbol{r})$ (charge per unit length) on these lines:

$$\beta\langle\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\frac{1}{2\pi^2}\left[\left(\frac{\pi}{2W}\right)^2\left(\frac{1}{\sinh\frac{\pi}{2W}(x-x')}\right)^2 - \left(\frac{1}{x-x'}\right)^2\right]$$

if $\boldsymbol{r}$ and $\boldsymbol{r}'$ are on the ideal conductor (2.3*a*)

$$\beta\langle\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\frac{1}{2}\left(\frac{1}{2W}\right)^2\left(\frac{1}{\cosh\frac{\pi}{2W}(x-x')}\right)^2$$

if $\boldsymbol{r}$ is on the ideal conductor and $\boldsymbol{r}'$ on the Coulomb system (2.3*b*)

and

$$\beta\langle\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\frac{1}{2\pi^2}\left[\left(\frac{\pi}{2W}\right)^2\left(\frac{1}{\sinh\frac{\pi}{2W}(x-x')}\right)^2 + \left(\frac{1}{x-x'}\right)^2\right]$$

if $\boldsymbol{r}$ and $\boldsymbol{r}'$ are on the Coulomb system. (2.3*c*)

((2.3*b*) and (2.3*c*) disregard the microscopic detail; (2.3*c*) must be regularized at $x - x' = 0$). As expected, if $W \to \infty$, the fluctuations (2.3*a*) on the ideal conductor C now alone in space disappear, as well as the correlations (2.3*b*) between the ideal conductor C and the Coulomb system S, while the fluctuations (2.3*c*) on the Coulomb system S become those of one conducting line [6].

From the averages $\langle E_\mu E_\nu\rangle$ associated to (2.2), one can compute the Maxwell stress tensor. Its only non-zero component is T_{yy}. At any point $\boldsymbol{r}$ between the lines $y = 0$ and $y = W$, assuming that there is no potential difference between the lines, one found [7]

$$\beta T_{yy} = \frac{\pi}{24W^2}. \tag{2.4}$$

$-T_{yy}$ is the force per unit length exerted on the Coulomb system S.

2.3. The high-density limit approach

We now assume that both lines $y = 0$ and $y = W$ are Coulomb systems, and the high-density limit is taken on the line $y = 0$. In that limit, the macroscopic approach [6] for computing the potential correlations is valid under the same conditions as in section 2.2: $\boldsymbol{r}$ and $\boldsymbol{r}'$ should be at macroscopic distances from the line $y = W$, but there is no restriction about their distances to the line $y = 0$. One finds

$$\beta\langle\Phi(\boldsymbol{r})\Phi(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\ln\left|z - \bar{z}'\right| \qquad \text{if} \quad (y, y') \in]-\infty, 0[^2 \tag{2.5*a*}$$

$$\beta\langle\Phi(\boldsymbol{r})\Phi(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\ln\left|(z - z')\frac{\sinh\frac{\pi}{2W}(z - \bar{z}')}{\sinh\frac{\pi}{2W}(z - z')}\right| \qquad \text{if} \quad (y, y') \in]0, W[^2 \tag{2.5*b*}$$

$$\beta\langle\Phi(\boldsymbol{r})\Phi(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\ln\left|z - z'\right| \qquad \text{if} \quad (y, y') \in]0, W[\times]W, +\infty[\tag{2.5c}$$

$$\beta\langle\Phi(\boldsymbol{r})\Phi(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\ln\left|z - \bar{z}' - \mathrm{i}W\right| \qquad \text{if} \quad (y, y') \in]W, +\infty[^2 \tag{2.5d}$$

$$\beta\langle\Phi(\boldsymbol{r})\Phi(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\ln\left|z - z'\right| \qquad \text{if} \quad (y, y') \in]-\infty, 0[\times]W, +\infty[. \tag{2.5e}$$

The difference between the electric potential correlations in this case and the previous case (equations (2.2)) is the electric potential correlation of a system where there is only one Coulomb system on the line $y = 0$ [6]. This difference arises from neglecting the fluctuations of the conductor at $y = 0$ in the ideal conductor case.

From (2.5) we obtain the charge correlations

$$\beta\langle\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\frac{1}{2\pi^2}\left[\left(\frac{\pi}{2W}\right)^2\left(\frac{1}{\sinh\frac{\pi}{2W}(x - x')}\right)^2 + \left(\frac{1}{x - x'}\right)^2\right] \qquad \text{if} \quad y = y' = 0 \tag{2.6a}$$

$$\beta\langle\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\frac{1}{2}\left(\frac{1}{2W}\right)^2\left(\frac{1}{\cosh\frac{\pi}{2W}(x - x')}\right)^2 \qquad \text{if} \quad y = 0 \quad \text{and} \quad y' = W \tag{2.6b}$$

$$\beta\langle\sigma(\boldsymbol{r})\sigma(\boldsymbol{r}')\rangle^{\mathrm{T}} = -\frac{1}{2\pi^2}\left[\left(\frac{\pi}{2W}\right)^2\left(\frac{1}{\sinh\frac{\pi}{2W}(x - x')}\right)^2 + \left(\frac{1}{x - x'}\right)^2\right] \qquad \text{if} \quad y = y' = W \tag{2.6c}$$

(Equations (2.6) disregard the microscopic detail; (2.6*a*) and (2.6*b*) must be regularized at $x - x' = 0$.)

The charge correlations in the Coulomb system at $y = W$ are the same in both approaches (equations (2.3*c*) and (2.6*c*)) as it was noticed before in other models [4, 2]. Also the correlation between a point on the conductor ($y = 0$) and a point on the Coulomb system ($y = W$) is the same in both approaches (equations (2.3*b*) and (2.6*b*)). But the correlation between two points on the conductor at $y = 0$ differ when the conductor is ideal (equation (2.3*a*)) and when it is the high-density limit of a Coulomb system (equation (2.6*a*)).

The stress tensor is the same as in equation (2.4). This shows that we have the same results in both approaches for the force exerted on the Coulomb system S. What is special in the present model and what is general will be discussed in section 3.

2.4. A solvable model

The charge correlations (2.3) and (2.6) can be checked on a solvable microscopic model. In this section we consider that the Coulomb system S is a one-component plasma: the system is composed of particles of charge q moving in a rigid charged background. The two-dimensional one-component plasma is a solvable model in several geometries [8–11] when $\beta q^2 = 2$.

(a) The system such that C is an ideal conductor has been solved in [12, 13, 7], in the grand canonical ensemble. Let $-q\eta$ be the background charge density of S and ζ the fugacity. The number density n and charge correlation in the Coulomb system S are given in terms of

$$g(x) = \int_0^\infty \frac{\mathrm{d}k}{2\pi} \frac{\mathrm{e}^{\mathrm{i}kx}}{1 + (2\pi\zeta)^{-1}\mathrm{e}^{2W(k-2\pi\eta)}} \tag{2.7}$$

as

$$n = g(0) \tag{2.8}$$

and

$$\langle \sigma(\boldsymbol{r})\sigma(\boldsymbol{r}')\rangle^{\mathrm{T}} = -q^2|g(x-x')|^2 + q^2 n\delta(x). \tag{2.9}$$

It has been shown in [7] that, in the macroscopic limit ($\eta W \gg 1$), these results agree with those from the macroscopic approach of section 2.2; (2.9) becomes (2.3*a*).

(b) The system such that C is a conducting Coulomb system can also be solved exactly. Now each line is a one-component plasma. The background charge densities are $-q\eta_0$ for the line $y = 0$ and $-q\eta$ for the line $y = W$. Here we work in the canonical ensemble. Let N be the total number of particles. We consider first that we have two concentric circles, with radii R and $R + W$, on which each plasma lies, and then take the limit $R \to \infty$, with $N = 2\pi\eta_0 R + 2\pi\eta(R + W)$, which ensures overall neutrality. Adapting [11] by treating the radial coordinate r as a discrete variable which can have the values R and $R + W$, we introduce the N orthogonal functions

$$\psi_\ell(\boldsymbol{r}) = \left(\alpha\ \delta_{r,R} + \delta_{r,r+W}\right) z^\ell \qquad 0 \leqslant \ell \leqslant N-1 \tag{2.10}$$

where $z = r\mathrm{e}^{\mathrm{i}\theta}$, δ is the Kronecker symbol, and α is a (positive) parameter which controls [14] how the N particles are distributed between the two lines. The density n and charge correlations are given in terms of the projector

$$P(\boldsymbol{r},\boldsymbol{r}') = \sum_{\ell=0}^{N-1} \frac{\Psi_\ell(\boldsymbol{r})\overline{\Psi_\ell(\boldsymbol{r}')}}{\sum_{r_0\in\{R,R+W\}} \int_0^{2\pi} |\Psi_\ell(r_0,\theta_0)|^2 r_0 \mathrm{d}\theta_0} \tag{2.11}$$

as

$$n(\boldsymbol{r}) = P(\boldsymbol{r},\boldsymbol{r}) \tag{2.12}$$

and

$$\langle \sigma(\boldsymbol{r})\sigma(\boldsymbol{r}')\rangle^{\mathrm{T}} = -q^2|P(\boldsymbol{r},\boldsymbol{r}')|^2 + q^2 n(\boldsymbol{r})\delta_{r,r'}\delta(r(\theta-\theta')). \tag{2.13}$$

In the limit $R \to \infty$ the two circles become two parallel lines. In this limit it is useful to define $k = \ell/R$. A summation over ℓ becomes an integral over k times R. We change our system of coordinates: let $x = R\theta$ and $y = r - R$. The projector becomes

$$P(\boldsymbol{r},\boldsymbol{r}') = \delta_{y,y'}(\delta_{y,0}P_1(x-x') + \delta_{y,W}P_2(x-x')) + (\delta_{y,0}\delta_{y',W} + \delta_{y,W}\delta_{y',0})P_3(x-x') \tag{2.14}$$

with

$$P_1(x) = \frac{1}{2\pi}\int_0^{2\pi(\eta_0+\eta)} \frac{\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k}{1+\alpha^{-2}\ \mathrm{e}^{2Wk}} \tag{2.15a}$$

$$P_2(x) = \frac{1}{2\pi}\int_0^{2\pi(\eta_0+\eta)} \frac{\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k}{1+\alpha^{2}\mathrm{e}^{-2Wk}} \tag{2.15b}$$

$$P_3(x) = \frac{1}{2\pi}\int_0^{2\pi(\eta_0+\eta)} \frac{\mathrm{e}^{\mathrm{i}kx}\mathrm{d}k}{\alpha\ \mathrm{e}^{-Wk}+\alpha^{-1}\mathrm{e}^{Wk}}. \tag{2.15c}$$

For a comparison with previous results, we define an alternative control parameter ζ by

$$2\pi\zeta = \alpha^{-2}\mathrm{e}^{4\pi\eta_0 W} \tag{2.16}$$

and keep ζ fixed as we vary the other parameters η_0, η, W.

When the density η_0 of the conductor C becomes infinite, from (2.15*b*) where we make the change of variable $k \to 2\pi(\eta_0+\eta)-k$, we obtain

$$|P_2(x)| = \left|\frac{1}{2\pi}\int_0^{\infty} \frac{\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}k}{1+(2\pi\zeta)^{-1}\ \mathrm{e}^{2W(k-2\pi\eta)}}\right|. \tag{2.17}$$

Since $|P_2(x)|$ as given by (2.17) is identical to $|g(x)|$ as given by (2.7), the density and the charge correlation function on the Coulomb system S are indeed identical whenever C is an ideal conductor or the high-density limit of a Coulomb system.

A more detailed comparison can be made when W is macroscopic. Then, neglecting terms $\exp(-\eta_0 W)$ and $\exp(-\eta W)$ in (2.15), after an averaging over oscillations of microscopic wavelength we obtain

$$|P_1(x)|^2 \sim |P_2(x)|^2 \sim \frac{1}{4\pi^2}\left[\left(\frac{1}{x}\right)^2 + \left(\frac{\pi}{2W\sinh\frac{\pi x}{2W}}\right)^2\right] \tag{2.18a}$$

$$|P_3(x)|^2 \sim \frac{1}{4\pi^2}\left(\frac{\pi}{2W\cosh\frac{\pi x}{2W}}\right)^2. \tag{2.18b}$$

The charge correlations obtained from (2.18) agree with the macroscopic ones obtained in (2.6).

3. General case

In this section we consider the general case in $d \geqslant 2$ dimensions. The conductor C has any shape and the Coulomb system S occupies some region of space outside C. The Coulomb potential is

$$G_0(\boldsymbol{r}) = \begin{cases} -\ln r & \text{if } d=2 \\ r^{2-d} & \text{if } d>2 \end{cases} \tag{3.1}$$

To start with, C itself is considered as a Coulomb system with internal degrees of freedom.

We shall use several quantities related to the electric potential correlations for the conductor C alone in space. Let $\Phi_c(\boldsymbol{r})$ be the electric potential at $\boldsymbol{r}$ created by the conductor C alone and let $\langle \ldots \rangle_0^T$ be a truncated statistical average computed with the Boltzmann weight of the conductor C alone. The correlation $\langle \Phi_c(\boldsymbol{r})\Phi_c(\boldsymbol{r}')\rangle_0^T$ can be computed by linear response [6]; it is related to the average electric potential change at $\boldsymbol{r}$ when a unit charge is put at $\boldsymbol{r}'$. This potential change can be computed by macroscopic electrostatics. If the conductor is grounded, there are two cases:

(a) If $\boldsymbol{r}$ (or $\boldsymbol{r}'$, or both) is (are) inside the conductor,

$$\beta\langle \Phi_c(\boldsymbol{r})\Phi_c(\boldsymbol{r}')\rangle_0^T = G_0(\boldsymbol{r}-\boldsymbol{r}') \tag{3.2a}$$

(b) If $\boldsymbol{r}$ and $\boldsymbol{r}'$ are outside the conductor,

$$\beta\langle \Phi_c(\boldsymbol{r})\Phi_c(\boldsymbol{r}')\rangle_0^T = G^*(\boldsymbol{r},\boldsymbol{r}') \tag{3.2b}$$

where G^* is defined by

$$\Delta_{\boldsymbol{r}}\left[G_0(\boldsymbol{r}-\boldsymbol{r}') - G^*(\boldsymbol{r},\boldsymbol{r}')\right] = -\mu_d\delta(\boldsymbol{r}-\boldsymbol{r}') \tag{3.3}$$

for $\boldsymbol{r}$ and $\boldsymbol{r}'$ outside the conductor with $\mu_2 = 2\pi$, $\mu_3 = 4\pi, \ldots, \mu_d = (d-2)2\pi^{d/2}/\Gamma(d/2)$ if $d > 2$, and the condition $G_0(\boldsymbol{r}-\boldsymbol{r}') - G^*(\boldsymbol{r},\boldsymbol{r}') = 0$ if $\boldsymbol{r}$ (or $\boldsymbol{r}'$) is on the surface of the conductor; $G_0(\boldsymbol{r}-\boldsymbol{r}') - G^*(\boldsymbol{r},\boldsymbol{r}')$ is the electric potential at $\boldsymbol{r}$ created by a unit charge at $\boldsymbol{r}'$ in the presence of a grounded ideal conductor. The expressions (3.2) which disregard the microscopic detail become exact in the limit when the correlation length goes to zero.

Another remark useful for the following sections is that the fluctuations of the electric potential of a conductor C are Gaussian [6, 15, 16].

Let $\boldsymbol{R}$ be the set of particle coordinates of the conductor C, $\{\boldsymbol{r}_i\}$ the set of the N particle coordinates of the Coulomb system S, $d\Gamma$ the element of phase space of the Coulomb system and $H_0(\boldsymbol{R})$ the Hamiltonian of the conductor. The total energy of the system (S plus C) is

$$H(\{\boldsymbol{r}_i\},\boldsymbol{R}) = \sum_{i=0}^{N} q_i\Phi_c(\boldsymbol{r}_i,\boldsymbol{R}) + \sum_{1\leqslant i<j\leqslant N} q_iq_jG_0(\boldsymbol{r}_i-\boldsymbol{r}_j) + H_0(\boldsymbol{R}) \tag{3.4}$$

where $\boldsymbol{r}_i$ and q_i are the position and charge of the ith particle of S. There might also be some short-range interaction between the particles of S, but we do not write it explicitly in (3.4) just to have a simpler notation.

In the following sections we shall compute the partition function, the correlations, the stress tensor and the force exerted on the Coulomb system S, in the limit when the charge correlation length of the conductor C goes to zero (we shall call this limit the good conductor limit) and compare the results to those when the conductor is ideal (i.e. the potential on it is fixed, say to zero, without fluctuations). For the sake of simplicity we shall only treat in detail this case of a grounded conductor, but similar results hold for an insulated conductor (see section 3.5).

3.1. Partition function and statistical averages

The partition function of the total system can be written as

$$Z = \int d\Gamma \langle e^{-\beta\sum_i q_i\Phi_c(\boldsymbol{r}_i)}\rangle_0 e^{-\beta\sum_{i<j} q_iq_jG_0(\boldsymbol{r}_i-\boldsymbol{r}_j)} Z_0 \tag{3.5}$$

where $\langle \ldots \rangle_0$ means the average over $\boldsymbol{R}$, with the Boltzmann weight $\exp(-\beta H_0(\boldsymbol{R}))$, and $Z_0 = \int \exp(-\beta H_0(\boldsymbol{R}))\mathrm{d}\boldsymbol{R}$ is the partition function of the conductor C alone.

Now, since the fluctuations of Φ_c are Gaussian

$$\left\langle \exp\left[-\beta \sum_i q_i \Phi_c(\boldsymbol{r}_i)\right]\right\rangle_0 = \exp\left[\frac{1}{2}\beta^2 \sum_{i,j} q_i q_j \langle \Phi_c(\boldsymbol{r}_i)\Phi_c(\boldsymbol{r}_j)\rangle_0^{\mathrm{T}}\right]$$
$$= \exp\left[\frac{1}{2}\beta \sum_{i,j} q_i q_j G^*(\boldsymbol{r}_i, \boldsymbol{r}_j)\right] \tag{3.6}$$

where we have used (3.2*b*). Thus, the partition function becomes

$$Z = Z^* Z_0 \tag{3.7}$$

where Z^* is the partition function of the Coulomb system S in the presence of an ideal conductor

$$Z^* = \int \mathrm{d}\Gamma \ \mathrm{e}^{-\beta H_{\mathrm{eff}}} \tag{3.8}$$

where

$$H_{\mathrm{eff}}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = -\frac{1}{2}\sum_{i=1}^{N} q_i^2 G^*(\boldsymbol{r}_i, \boldsymbol{r}_i) + \sum_{1 \leqslant i < j \leqslant N} q_i q_j \left[G_0(\boldsymbol{r}_i - \boldsymbol{r}_j) - G^*(\boldsymbol{r}_i, \boldsymbol{r}_j)\right]. \tag{3.9}$$

H_{eff} is indeed the standard Hamiltonian used in the ideal conductor approach. For instance, in the case of a plane ideal conductor, G^* is the particle/image interaction; it should be noted that the interaction $-q_i^2 G^*(\boldsymbol{r}_i, \boldsymbol{r}_i)$ of a particle with its own image carries a factor $\frac{1}{2}$ in (3.9).

The total free energy is $F = F^* + F_0$ where F^* is the free energy of S in the presence of an ideal conductor and F_0 the free energy of the conductor C alone. This was noticed previously in [4] for the model of a two-dimensional plasma near a metallic wall.

Let $A(\{\boldsymbol{r}_i\})$ be a microscopic quantity that does not depend on $\boldsymbol{R}$. Its thermodynamic average, by (3.6), is

$$\begin{aligned}\langle A \rangle &= \frac{1}{Z^*}\int \mathrm{d}\Gamma \langle \mathrm{e}^{-\beta \sum_i q_i \Phi_c(\boldsymbol{r}_i)}\rangle_0 \mathrm{e}^{-\beta \sum_{i<j} q_i q_j G_0(\boldsymbol{r}_i - \boldsymbol{r}_j)} A \\ &= \frac{1}{Z^*}\int \mathrm{d}\Gamma \mathrm{e}^{-\beta H_{\mathrm{eff}}(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N)} A \\ &= \langle A \rangle_{\mathrm{eff}}. \end{aligned} \tag{3.10}$$

Thus the average of A can be computed by assuming from the beginning that the conductor C is ideal.

3.2. Electric potential correlations

Equation (3.10) does not apply to the electric potential correlations because the microscopic electric potential is different in the cases of a good conductor or an ideal conductor. For the good conductor case, the microscopic electric potential is

$$\Phi(\boldsymbol{r}) = \sum_i q_i G_0(\boldsymbol{r} - \boldsymbol{r}_i) + \Phi_c(\boldsymbol{r}) \tag{3.11}$$

while for the ideal conductor case it is

$$\Phi_{id}(\boldsymbol{r}) = \begin{cases} 0 & \text{if } \boldsymbol{r} \in C \\ \sum_i q_i \left[G_0(\boldsymbol{r} - \boldsymbol{r}_i) - G^*(\boldsymbol{r}, \boldsymbol{r}_i) \right] & \text{if } \boldsymbol{r} \notin C. \end{cases} \tag{3.12}$$

The average electric potential in the good conductor case is

$$\langle \Phi(\boldsymbol{r}) \rangle = \frac{1}{Z^*} \int d\Gamma \Big\{ \sum_i q_i G_0(\boldsymbol{r} - \boldsymbol{r}_i) e^{-\beta H_{eff}} + \langle \Phi_c(\boldsymbol{r}) e^{-\beta \sum_i q_i \Phi_c(\boldsymbol{r}_i)} \rangle_0 e^{-\beta \sum_{i<j} q_i q_j G_0(\boldsymbol{r}_i - \boldsymbol{r}_j)} \Big\}. \tag{3.13}$$

Since the fluctuations of Φ_c are Gaussian,

$$\langle \Phi_c(\boldsymbol{r}) e^{-\beta \sum_i q_i \Phi_c(\boldsymbol{r}_i)} \rangle_0 = -\beta \sum_i q_i \langle \Phi_c(\boldsymbol{r}) \Phi_c(\boldsymbol{r}_i) \rangle_0 e^{\frac{1}{2}\beta^2 \sum_{i,j} q_i q_j \langle \Phi_c(\boldsymbol{r}_i) \Phi_c(\boldsymbol{r}_j) \rangle_0}. \tag{3.14}$$

Using the covariance (3.2), one finds, for all $\boldsymbol{r}$

$$\langle \Phi(\boldsymbol{r}) \rangle = \langle \Phi_{id}(\boldsymbol{r}) \rangle_{eff} \tag{3.15}$$

where $\langle \Phi_{id}(\boldsymbol{r}) \rangle_{eff}$ is the average electric potential in the ideal conductor case.

We can compute the electric potential correlations in the same way: the correlation function in the good conductor case can be written as

$$\begin{aligned} \beta \langle \Phi(\boldsymbol{r}) \Phi(\boldsymbol{r}') \rangle = \frac{1}{Z^*} \int d\Gamma \Big\{ & \sum_{i,j} q_i q_j G_0(\boldsymbol{r} - \boldsymbol{r}_i) G_0(\boldsymbol{r}' - \boldsymbol{r}_j) e^{-\beta H_{eff}} \\ & + e^{-\beta \sum_{i<j} q_i q_j G_0(\boldsymbol{r}_i - \boldsymbol{r}_j)} \Big[\sum_i q_i G_0(\boldsymbol{r} - \boldsymbol{r}_i) \langle \Phi_c(\boldsymbol{r}') e^{-\beta \sum_i q_i \Phi_c(\boldsymbol{r}_i)} \rangle_0 \\ & + \sum_i q_i G_0(\boldsymbol{r}' - \boldsymbol{r}_i) \langle \Phi_c(\boldsymbol{r}) e^{-\beta \sum_i q_i \Phi_c(\boldsymbol{r}_i)} \rangle_0 \\ & + \langle \Phi_c(\boldsymbol{r}) \Phi_c(\boldsymbol{r}') e^{-\beta \sum_i q_i \Phi_c(\boldsymbol{r}_i)} \rangle_0 \Big] \Big\}. \end{aligned} \tag{3.16}$$

Using (3.15)

$$\begin{aligned} \langle \Phi_c(\boldsymbol{r}) \Phi_c(\boldsymbol{r}') e^{-\beta \sum_i q_i \Phi_c(\boldsymbol{r}_i)} \rangle_0 = & \Big[\beta^2 \sum_{i,j} q_i q_j \langle \Phi_c(\boldsymbol{r}) \Phi_c(\boldsymbol{r}_i) \rangle_0 \langle \Phi_c(\boldsymbol{r}') \Phi_c(\boldsymbol{r}_j) \rangle_0 \\ & + \langle \Phi_c(\boldsymbol{r}) \Phi_c(\boldsymbol{r}') \rangle_0 \Big] e^{\frac{1}{2}\beta^2 \sum_{i,j} q_i q_j \langle \Phi_c(\boldsymbol{r}_i) \Phi_c(\boldsymbol{r}_j) \rangle_0} \end{aligned} \tag{3.17}$$

(also a consequence of Φ_c being Gaussian), and the covariance (3.2), we find

$$\langle \Phi(\boldsymbol{r}) \Phi(\boldsymbol{r}') \rangle^T = \langle \Phi_{id}(\boldsymbol{r}) \Phi_{id}(\boldsymbol{r}') \rangle^T_{eff} + \langle \Phi_c(\boldsymbol{r}) \Phi_c(\boldsymbol{r}') \rangle^T_0. \tag{3.18}$$

Thus, the correlation function in the presence of a good conductor is the correlation function in the presence of an ideal conductor plus the correlation function for the good conductor alone in space.

This is what was noticed in the example of section 2.

3.3. Charge correlations

If we are interested in charge correlations in the Coulomb system S, equation (3.10) applies because the microscopic charge density outside the conductor

$$\rho(\boldsymbol{r}) = \sum_{i=1}^{N} q_i \delta(\boldsymbol{r} - \boldsymbol{r}_i) \tag{3.19}$$

does not depend on the coordinates $\boldsymbol{R}$; thus the charge correlations inside S are the same in both approaches. The surface charge density on C is given by the discontinuity of the normal electric field, thus using (3.15) we find that the average charge density is the same in both approaches. The same holds for the correlation between the density in S and the surface charge density on C. But, if we are interested in charge correlations on the conductor C, the correlations are different in the two approaches. Using equation (3.18) we can compute the difference in the electric field correlations and from it the difference in the charge correlations on the surface of C; this difference is the surface charge correlation on C when it is alone in space.

The example of section 2 illustrates these general results.

3.4. The stress tensor and the forces exerted on the Coulomb system

The Maxwell stress tensor is

$$T_{\mu\nu} = \mu_d^{-1} \left\langle E_\mu E_\nu - \frac{\delta_{\mu\nu}}{2} \boldsymbol{E}^2 \right\rangle \tag{3.20}$$

where $\boldsymbol{E} = -\nabla\Phi$ is the electric field. Let $\mathcal{V}$ be some volume outside the conductor C. The total average electric force on $\mathcal{V}$ is

$$\boldsymbol{F} = \int_{\partial\mathcal{V}} \boldsymbol{T} \cdot \mathrm{d}^{d-1}\boldsymbol{S} = \int_{\mathcal{V}} \nabla \cdot \boldsymbol{T}(\boldsymbol{r}) \mathrm{d}^d \boldsymbol{r}. \tag{3.21}$$

It can be shown that this force is the same in both models although the electric potential correlations are different and consequently the stress tensor might be different. Indeed, from equation (3.18), the difference between the stress tensor in the good conductor case and the ideal conductor case is

$$T_{\mu\nu}(\boldsymbol{r}) - T^{\mathrm{id}}_{\mu\nu}(\boldsymbol{r}) = \mu_{\mathrm{d}}^{-1} \left(\partial_\mu \partial'_\nu - \frac{\delta_{\mu\nu}}{2} \partial_\sigma \partial'_\sigma \right) G^*(\boldsymbol{r}, \boldsymbol{r}) \tag{3.22}$$

where ∂ (respectively ∂') means partial differentiation with respect to the first (second) argument of G^*. The difference of the divergences is

$$\partial_\mu \left[T_{\mu\nu}(\boldsymbol{r}) - T^{\mathrm{id}}_{\mu\nu}(\boldsymbol{r}) \right] = \mu_d^{-1} \left(\partial'_\nu \partial_\mu \partial_\mu G^*(\boldsymbol{r}, \boldsymbol{r}) + \tfrac{1}{2} \partial_\mu \partial'_\mu \left[\partial'_\nu G^*(\boldsymbol{r}, \boldsymbol{r}) - \partial_\nu G^*(\boldsymbol{r}, \boldsymbol{r}) \right] \right). \tag{3.23}$$

Now, from (3.3) $\partial_\mu \partial_\mu G^*(\boldsymbol{r}, \boldsymbol{r}) = 0$, and since G^* is symmetrical $\partial'_\nu G^*(\boldsymbol{r}, \boldsymbol{r}) = \partial_\nu G^*(\boldsymbol{r}, \boldsymbol{r})$. Thus, (3.23) is zero and the force is the same in both models.

In the example of section 2, the stress tensor itself was the same in the presence of either a good conductor or an ideal conductor. But this was an effect of the very peculiar symmetry of the model (invariance by translations along the x axis).

A direct evaluation of the average force exerted on one particle of the system confirms that the two approaches give the same result. Let $d\Gamma_{N-1}$ be the phase space element of the system S when the ith particle is fixed and Z^*_{N-1} the partition function of the system S in presence of an ideal conductor when the ith particle is fixed. The average force exerted on the ith particle in presence of a good conductor is

$$\langle \boldsymbol{F}_i \rangle = -\frac{1}{Z^*} \int d\Gamma_{N-1} \left\{ \nabla_i \left[\sum_{l<k} q_l q_k G_0(\boldsymbol{r}_l - \boldsymbol{r}_k) \right] e^{-\beta H_{\text{eff}}} + \left\langle \nabla_i \left[\sum_l q_l \Phi_c(\boldsymbol{r}_l) \right] e^{-\beta \sum_k q_k \Phi_c(\boldsymbol{r}_k)} \right\rangle_0 e^{-\beta \sum_{l<k} q_l q_k G_0(\boldsymbol{r}_l - \boldsymbol{r}_k)} \right\}. \tag{3.24}$$

In the last term

$$\begin{aligned} \left\langle \nabla_i \left[\sum_l q_l \Phi_c(\boldsymbol{r}_l) \right] e^{-\beta \sum_k q_k \Phi_c(\boldsymbol{r}_k)} \right\rangle_0 &= -\beta^{-1} \nabla_i \left[\langle e^{-\beta \sum_k q_k \Phi_c(\boldsymbol{r}_k)} \rangle_0 \right] \\ &= -\beta^{-1} \nabla_i \left[e^{\frac{\beta}{2} \sum_{kl} q_k q_l G^*(\boldsymbol{r}_k, \boldsymbol{r}_l)} \right] \\ &= -\nabla_i \left[\sum_{kl} \frac{q_k q_l}{2} G^*(\boldsymbol{r}_k, \boldsymbol{r}_l) \right] e^{\frac{\beta}{2} \sum_{kl} q_k q_l G^*(\boldsymbol{r}_k, \boldsymbol{r}_l)}. \end{aligned} \tag{3.25}$$

Using (3.25) in (3.24) gives

$$\langle \boldsymbol{F}_i \rangle = \langle \boldsymbol{F}_i^{\text{id}} \rangle_{\text{eff}}. \tag{3.26}$$

We obtain the same average force by both approaches.

3.5. The insulated conductor case

In the former calculations we assumed that the conductor C was grounded. The same calculations can be carried out if the conductor C is insulated. Equations (3.7), (3.10), (3.15), (3.26) still hold for the insulated conductor case. The expression of H_{eff} now is a different one but it is still the Hamiltonian of the system S in the presence of an ideal conductor: everywhere $G^*(\boldsymbol{r}, \boldsymbol{r}')$ must be replaced by $G^*(\boldsymbol{r}, \boldsymbol{r}') + Q(\boldsymbol{r})\ V(\boldsymbol{r}')$ where $Q(\boldsymbol{r})$ is the charge created by the influence of a unit charge at $\boldsymbol{r}$ on the grounded conductor and $V(\boldsymbol{r}')$ is the electric potential at $\boldsymbol{r}'$ created by the conductor carrying a unit charge. The important fact to notice is that the relation (3.18) between the different potential correlation functions is still valid.

4. Conclusion

Two different methods for treating the problem of a Coulomb system near a conductor have been compared.

The first method, where the conductor is considered from the beginning as ideal, neglects all fluctuations in the conductor. The second method treats the conductor as a conducting Coulomb system the charge correlation length of which goes to zero. Even in that limit, the fluctuations in the conductor do not vanish. This modifies the electric potential correlations by a term given by the electric potential correlation when the conductor is alone, but the

average electric potential is not modified. Because of this, quantities such as the electric field and charge density will have the same properties: their average is the same in both methods but their correlations differ by the correlation for the conductor alone. However, in the case of the charge correlations, this difference vanishes outside the conductor.

The free energy in the good conductor case is just the sum of the free energy in the ideal conductor case plus the free energy when the conductor is alone in space.

The average stress tensor is modified only by a term the divergence of which is zero, so the average force exerted on any part of the Coulomb system is not modified. However, the fluctuations of these forces will in general be different in the two methods.

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